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A Legendre-pseudospectral method for computing travelling waves with corners (slope discontinuities) in one space dimension with application to Whitham's equation family

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Abstract

If the dispersion in a nonlinear hyperbolic wave equation is weak in the sense that the frequency $\omega(k)$ of $\cos(kx)$ is bounded as $k \to \infty$, it is common that (i) travelling waves exist up to a limiting amplitude with wave-breaking for higher amplitudes, and (ii) the limiting wave has a *corner*, that is, a discontinuity in slope. Because "corner" waves are not smooth, standard numerical methods converge poorly as the number of grid points is increased. However, the corner wave is important because, at least in some systems, it is the *attractor* for all large amplitude initial conditions. Here we devise a Legendre-pseudospectral method which is uncorrupted by the singularity. The symmetric ($u(X) = u(-X)$) wave can be computed on an interval spanning only half the spatial period; since u is smooth on this domain which does not include the corner except as an endpoint, all numerical difficulties are avoided. A key step is to derive an extra boundary condition which uniquely identifies the corner wave. With both the grid point values of $u(x)$ and phase speed c as unknowns, the discretized equations, imposing three boundary conditions on a second order differential equation, are solved by a Newton–Raphson iteration. Although our method is illustrated by the so-called "Whitham's equation", $u_t + uu_x = \int \mathscr{D}u \, dx'$ where $\mathscr{D}$ is a very general linear operator, the ideas are widely applicable.

Keywords: Corner waves; Cnoidal waves; Pseudospectral algorithm; Whitham equation

1. Introduction

In a study of equatorially trapped Kelvin waves [13], we found that travelling waves, as predicted by the Korteweg-deVries equation-based theory of [3], did indeed occur provided the initial amplitude was

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sufficiently small. However, in contradiction to Korteweg-deVries theory, which allows solitary and cnoidal waves of *arbitrarily large* amplitude, sufficiently tall Kelvin waves invariably steepen and break. Fig. 10 of [13] charts the boundary between non-breaking and breaking behavior for an initially sinusoidal disturbance of various amplitudes and wavelengths. Later work (to be published) has shown that the largest amplitude Kelvin travelling wave seems to have a slope discontinuity at the crest, that is, the shape of the wave has a corner.

Pullin's review [21] notes on page 109 that "solution branches for many families of vortex equilibria terminate where the conotur shape forms a 90° corner,..., examples being V-states with $m \geqslant 3$ and waves of finite amplitude on wall-bounded vortex layers [9]". A similar pattern of small amplitude, spatially periodic travelling waves, a limiting wave of discontinuous slope, and breaking for larger amplitude is found in ordinary non-rotating surface water waves. Sir George Stokes in 1847 showed that the sides of the crest of the limiting wave met at an angle of 120° (see [27, Appendix B]). With this century-old exemplar, one might imagine that the theory of weakly dispersive waves that exhibit similar break-at-large/smooth-at-small/corner wave limit behavior would be thoroughly developed.

Instead, as noted by Shefter and Rosales [26], it is only in the last five years that there has been much interest in corner waves, despite some pioneering efforts reviewed in [20,31], and vast numbers of questions are unresolved. In this note, we shall focus on a generalization of the one-dimensional advection equation and the Korteweg-deVries equation, dubbed "Whitham's equation" in [15]

$$(u_t + uu_x)_x = \mathscr{D}u \quad \text{[Whitham's Equation Family]}, \tag{1}$$

where the subscripts denote differentiation with respect to the subscripted coordinate and where $\mathscr{D}$ is a linear operator that we will dub the "dispersion operator" since this term is solely responsible for wave dispersion. Although we shall pursue this generalized wave equation family strictly as a mathematical model, it is not in any sense a made-up problem (although its original invention by Whitham [30,31] was rather heuristic). Rather, it falls naturally out of singular perturbation theory for a wide variety of problems in fluid mechanics and other branches of science and engineering with various $\mathscr{D}$ arising in different parameter ranges. Important special cases that have been previously studied include the following:

1. $\mathscr{D}u = u_{xxxx}$, Korteweg-deVries equation [3,6,31];
2. $\mathscr{D}u = u_{6x}$, Fifth-Order Korteweg-deVries equation [4];
3. $\mathscr{D}u = u$, Ostrovsky–Hunter equation [16,17];
4. $\mathscr{D}u = \int_0^{2\pi} \cos(x-y)u(y)\,\mathrm{d}y$, Gabov–Shefter–Rosales equation [15,26];
5. $$\mathscr{D}u = pb^2\left\{u - \int_0^{2\pi} \frac{b\cosh(b\{|X-y|-\pi\})}{2\sinh(\pi b)}u(y)\,\mathrm{d}y\right\},$$

   Whitham–Zaitsev kernel [20,25,30–32]

Many other cases are discussed in Whitham's book [31] and especially in the monograph by Naumkin and Shishmarev [20]. The operator can be generalized to include dissipative terms [18,20], but we shall restrict $\mathscr{D}$ to be purely dispersive.

Fig. 1 shows the travelling waves and the limiting corner wave for a typical case, the Ostrovsky–Hunter equation. Fig. 2 compares the corner waves for several representative cases.

We shall further restrict our goals to that of devising a numerical method to compute corner waves directly, employing only ideas that can be generalized to more complicated wave equations. These corner waves are more important than merely a limit of the travelling waves. For the wave equations studied by Shefter and Rosales [26] and Madja et al. [19], large amplitude initial conditions break and dissipate, but the waves do not decay to zero, but rather *evolve* to the *corner wave*. In other words, when small dissipative terms are added (to avoid numerical disaster at the shocks and also because all real fluids are viscous), the corner wave is an *attractor*.

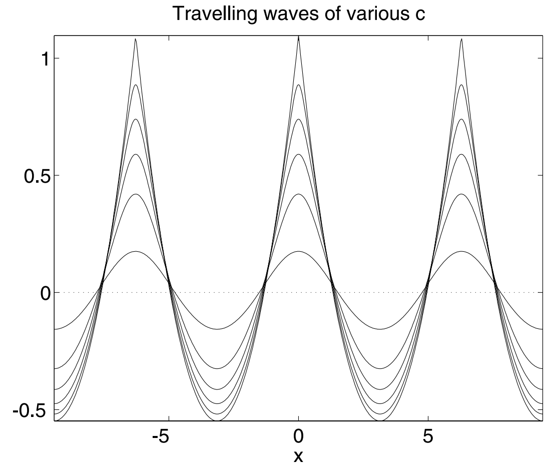

Fig. 1. Travelling waves and the corner wave for a typical weakly dispersive equation, Ostrovsky–Hunter equation. There are no travelling waves whose amplitudes are larger than that of the slope-discontinuous corner wave. The small amplitude travelling waves are closely approximated by a constant times $\cos(X)$.

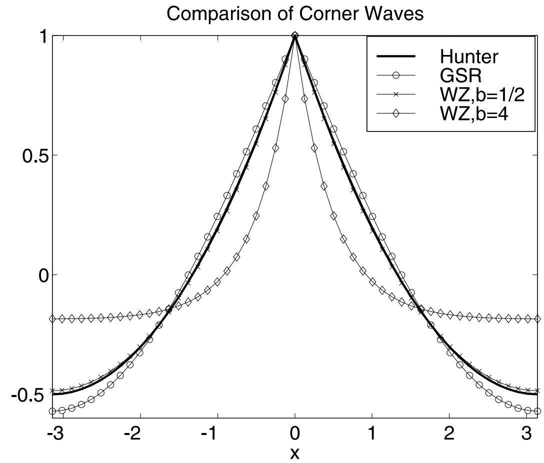

Fig. 2. A comparison of four typical corner waves: Ostrovsky–Hunter equation (thick solid), the Gabov–Shefter–Rosales equation with a cosine kernel (circles), and the Whitham–Zaitsev kernel for two different values of the parameter b which appears in the definition of this kernel. The amplitudes have all been scaled to a maximum of one to facilitate comparisons; in particular, the profiles of the Ostrovsky–Hunter and Gabov–Shefter–Rosales corner waves closely resemble the Whitham–Zaitsev corner wave for $b = 1/2$. In the limit $b \to \infty$, Zaitsev's periodic corner waves become the solitary waves analyzed by Whitham, Seliger and Gabov.

There are no rigorous theorems that separate classes of dispersion operator that generate travelling waves of arbitrary amplitude from classes which "max out" with a corner wave. However, experience with the special cases enumerated above and others not listed suggests that, in Whitham's equation family and in other wave equations, corner waves arise only when the dispersion is weak in the following sense.

Definition 1.1 (*Weak dispersion*). The dispersion operator $\mathscr{D}$ is said to generate weak dispersion if

$$\lim_{k\to\infty} \mathscr{D} \cos(kx) < \infty. \tag{2}$$

In what follows, we shall restrict attention to operators that are "weakly dispersive" in this sense. For steadily translating waves, which by definition are functions only of

$$X \equiv x - ct. \tag{3}$$

Whitham's equation family reduces to

$$(u - c)u_{XX} + (u_X)^2 = \mathscr{D}u \tag{4}$$

which is the problem we shall attack in the rest of the article. By rescaling the spatial coordinate, one can always choose the period to be 2π without loss of generality, and we shall do this.

In the next section, we shall review some needed theoretical background, and then derive the three boundary conditions that we will impose on (4) to simultaneously determine the unique $u(X;c)$ and c so that the wave is a corner wave.

Section 3 describes our Legendre-pseudospectral numerical method with Newton–Raphson iteration and continuation. The following section describes a numerical example. Next, we collect the three cases where the corner waves have been found in explicit analytic form by previous studies, and furnish the travelling "coshoidal" waves. The final section discusses the many open problems.

2. Background theory and derivation of boundary/matching conditions

The dispersion operator can always be represented in the form of a convolution operator

$$\mathscr{D}u = \frac{1}{\pi}\int_0^{2\pi} K(X - y)u(y)\,\mathrm{d}y. \tag{5}$$

The kernel is not necessarily a smooth function; $K(X)$ must be the derivative of a delta-function in order to yield derivative operators as needed for the Korteweg-deVries equation.

We shall assume the following:

1. $u(X)$ and the kernel K are spatially periodic;
2. the spatial period is 2π;
3. the corner is located at $X = 0$;
4. $u(X)$ and the kernel $K(X)$ are *symmetric* with respect to X, that is,

$$u(X) = u(-X) \tag{6}$$

and similarly for the kernel.

By rescaling the spatial coordinate, we can always convert a general period P into a period of 2π, so no generality is lost by our second assumption.

Whitham's equation family has coefficients which are independent of x, and therefore the solutions are always *translationally invariant* in the sense that if $u(x,t)$ is a solution, then so is $u(x+\phi,t)$ where ϕ is an arbitrary constant. For the corner waves, this implies that no generality is lost by assuming that the wave has been translated so that the slope discontinuity is always at $X = 0$ (third assumption).

Unfortunately, there is no rigorous proof that the corner waves must be symmetric with respect to their crest or trough. However, all known examples are symmetric. We shall therefore *assume* that the corner waves are symmetric with respect to $X = 0$.

It is helpful to prove the following:

Theorem 2.1 (Boundedness of $\mathscr{D}u$ for weak dispersion). *"Weak" dispersion, as defined above, is equivalent to the boundedness of $\mathscr{D}u$ even at a point where $u(X)$ has a discontinuous first derivative.*

Proof. Because of their symmetry and periodicity, both $K(X)$ and $u(X)$ can be expanded as cosine series:

$$K(X) \equiv \sum_{n=0}^{\infty} K_n \cos(nX), \quad u(X) \equiv \sum_{m=0}^{\infty} a_m \cos(mX). \tag{7}$$

From the definition of the dispersion operator plus the identity $\cos(n[X-y]) = \cos(nX)\cos(ny) + \sin(nX)\sin(ny)$ and the orthogonality of the trigonometric functions, we obtain

$$\mathscr{D}u = \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} K_n a_m \int_0^{2\pi} \cos(n[X-y])\cos(my)\,\mathrm{d}y, \tag{8}$$

$$= \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} K_n a_m \left\{ \cos(nX) \int_0^{2\pi} \cos(ny)\cos(my)\,\mathrm{d}y + \sin(nX) \int_0^{2\pi} \sin(ny)\cos(my)\,\mathrm{d}y \right\}$$

$$= 2K_0 a_0 + \sum_{n=1}^{\infty} K_n a_n. \tag{9}$$

Our definition of weak dispersion is that $\mathscr{D}\cos(nX)$ is finite as $n \to \infty$, which clearly implies that $|K_n|$ is bounded by a constant $\kappa > 0$. If the series for $u(X)$ is convergent, then so also will be the series for $\mathscr{D}u$, implying that $\mathscr{D}u$ is finite for all real X including at the point of discontinuous slope. (Note that the Fourier cosine coefficients of a function with a slope discontinuity are proportional to $1/n^2$ as $n \to \infty$.) □

The first boundary condition on $u(X)$ comes from the following:

Theorem 2.2. *If $\mathscr{D}$ is a "weak" dispersion operator, then for corner waves at the corner $X = 0$ where the first derivative of u has a jump discontinuity and the second derivative is a delta-function*

$$u(0) = c. \tag{10}$$

Proof. $\mathscr{D}u$ is bounded and smooth at the corner as shown by Theorem 2.1. In the theory of distributions, a jump discontinuity is proportional to the step function, and the derivative of the step function is the Dirac delta-function. Thus, the corner is a point where u_{XX} is proportional to $\delta(X)$. It follows that the left-hand side of Whitham's equation family cannot be smooth, matching the smoothness of the right-hand side and allowing the differential equation to be satisfied, unless $(u - c) = 0$ at the corner so as to multiply by zero the delta-function in u_{XX}. □

Theorem 2.3 (Dispersion operator has zero mean). *In order that the differential equation can be satisfied, it is necessary that*

$$\int_0^{2\pi} \mathscr{D}u(X)\,\mathrm{d}x = 0 \tag{11}$$

whether $u(X)$ is a corner wave or a smooth travelling wave of period 2π.

Proof. The differential equation is

$$(u - c)u_{XX} + (u_X)^2 = \mathscr{D}u. \tag{12}$$

However, the left-hand side is a perfect derivative so that the differential equation can be rewritten as

$$\{(u - c)u_X\}_X = \mathscr{D}u. \tag{13}$$

Integrating both sides gives

$$\{(u(2\pi) - c)u_X(2\pi)\} - \{(u(0) - c)u_X(0)\} = \int_0^{2\pi} \mathscr{D}u(X)\,\mathrm{d}X. \tag{14}$$

If we invoke the previous theorem that $u(0) = c$, and by periodicity, $u(2\pi) = c$ also, and also note that u_X is bounded at the corners, then the terms on the left-hand side of the integrated equation are zero. Therefore, the integral of $\mathscr{D}u$ must be zero also. If the wave does not have corners, then we can simply invoke spatial periodicity with period 2π to show that the terms on the left of the integrated equation, although not necessarily zero, must cancel.

Theorem 2.4 (Zero mean of solution). *If $u(X; c = c_0)$ is a solution to the generalized travelling wave equation and the kernel has zero mean, that is,*

$$K_0 \equiv \int_0^{2\pi} K(X)\,\mathrm{d}X = 0, \tag{15}$$

then $\mu + u(X; c_0)$ is a solution with $c = c_0 + \mu$. In this case, there is no loss of generality in assuming the zero mean condition

$$a_0 \equiv \int_0^{2\pi} u(X)\,\mathrm{d}X = 0 \quad [\textit{Zero Mean Condition}]. \tag{16}$$

If $K_0 \neq 0$, then the travelling must have zero mean.

Proof. For the first proposition where $K_0 = 0$, note that if we simultaneously replace u by $u + \lambda$ and c by $c + \lambda$, the left-hand side of the wave equation, $(u - c)u_{XX} + (u_X)^2$, is unaltered. The right-hand side is changed by $2\lambda K_0$, which is zero whenever the kernel has zero mean. To prove the second proposition, observe that (8) shows that the mean of $\mathscr{D}u$ is twice the product of the mean of K times the mean of u. However, an earlier theorem shows that the mean of $\mathscr{D}$ is zero. It follows that if $K_0 \neq 0$, then the mean of u must be zero. □

This theorem implies that we can restrict attention to solutions with zero mean without loss of generality, and we shall therefore assume (16) in the rest of the article.

The boundary condition that $u_X(\pi) = 0$ is a consequence of the following.

Theorem 2.5 (Double parity of periodic functions). *If $u(X)$ is a function of definite parity with respect to the origin and period 2π, then it is of the same parity with respect to the "half-period points", $X = \pm\pi$, also.*

Proof. A periodic, symmetric function may be expanded as a Fourier cosine series. By using the trigonometric identity, $\cos(n[X + \pi]) = (-1)^n \cos(nX)$, it follows that the function can be expanded as a series of cosines in the shifted coordinate $Y = X - \pi$ also. This implies that $u(X)$ must be symmetric about $X = \pi$. The same is true for antisymmetric functions, which may be expanded as sine series about both $X = 0$ and $X = \pi$. □

If $u(X)$ is smooth about a point of symmetry, then u_X is zero at this point. This follows from the usual centered definition of a derivative: $u_X = \lim_{h\to 0}(u(X + h) - u(X - h))/(2h)$, but $u(h) = u(-h)$ for a symmetric function for all h.

The symmetry is not merely important for deriving a boundary condition, however. Because of the symmetry, it is sufficient to numerically solve the integro-differential equation on the interval $X \in [0, \pi]$ even though the spatial period is twice as large. This greatly reduces the computational expense.

The final boundary condition can be derived by taking the limit of the terms in the differential equation as $X \to 0+$. The dispersive term $\mathscr{D}u$ does not blow up even at the corner – this is the meaning of "weak" dispersion according to Theorem 2.1 – and tends smoothly to its limit as $X \to 0$. The theorem $u(0) = c$ implies that the $(u - c)u_{XX}$ is approximately zero for very small but non-zero X. The differential equation in the limit is

$$(u_X(0))^2 = \mathscr{D}u(0). \qquad \square \tag{17}$$

The same limit may also be derived [8] using matched asymptotic expansions [1,2,23,28].

3. Numerical implementation

The numerical problem is to solve

$$(u - c)u_X + (u_X)^2 = \mathscr{D}u, \quad X \in [0, \pi] \tag{18}$$

with c as an unknown as well as u, subject to the three boundary conditions:

$$u_X(\pi) = 0, \quad u(0) = c, \quad (u_X(0))^2 = \mathscr{D}u(0). \tag{19}$$

Because we assume that the corner wave is *symmetric* with respect to $X = 0$, it is sufficient to restrict the numerical domain to half the spatial period. Because $u(X)$ is completely smooth on the half-domain, it has a rapidly convergent series as a Legendre polynomial series.

We chose a pseudospectral method because it combines high accuracy with simplicity of programming. Pseudospectral algorithms in general are described at length in the monographs [7,12] and the review [22].

One minor technical complication is that the canonical interval for Legendre expansions is $z \in [-1, 1]$. It is therefore convenient to make the change of variable

$$z = -1 + \frac{2}{\pi}X \leftrightarrow X = \frac{\pi}{2}(z+1) \to \frac{\mathrm{d}}{\mathrm{d}X} = \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}z}, \quad \mathrm{d}X = \frac{\pi}{2}\,\mathrm{d}z. \tag{20}$$

The problem becomes

$$(u - c)u_{zz} + (u_z)^2 = \frac{\pi^2}{4}\mathscr{D}u, \quad z \in [-1, 1]. \tag{21}$$

The first two boundary conditions are essentially unaltered, i.e., $u_z(1) = 0$, $u(z = -1) = c$ while the third boundary condition is modified to $[u_z(z = -1)]^2 = (\pi^2/4)\mathscr{D}u(z = -1)$. Because of the symmetry of $u(X)$ with respect to $X = \pi$, the convolution integral (5) can be written as

$$\mathscr{D}u = \frac{1}{2}\int_{-1}^{1}\left\{K\left(\frac{\pi}{2}(z - w)\right) + K\left(\frac{\pi}{2}(z + w - 2)\right)\right\}u\left(\frac{\pi}{2}(w+1)\right)\mathrm{d}w. \tag{22}$$

When the operator $\mathscr{D}u$ is thus represented as an integral, a basis of Legendre polynomials is convenient; the collocation points associated with these functions are also the abcissas for a Gauss–Lobatto quadrature. The collocation/quadrature points z_i are the N roots of $(1 - z^2)P_{N-1,z}(z)$, where $P_{N-1,z}$ denotes the first derivative of the Legendre polynomial of degree $N - 1$. The Legendre–Lobatto points for up to nine-point grids are given analytically on pages 572–574 of [7]. For arbitrary N, one can use the Fortran software given in Appendix C of [12] or the Matlab codes of [29].

Instead of using the Legendre polynomials themselves as the basis set ("modal basis"), it is convenient to rearrange the first N polynomials into polynomials C_j of degree N which have the property of being one at

z_j and zero at all the other collocation points. The elements of this "nodal" or "Lagrange" or "cardinal" basis are

$$C_j(z) \equiv -\frac{1-z^2}{N(N-1)P_{N-1}(z_j)(z-z_j)}\frac{\mathrm{d}P_{N-1}(z)}{\mathrm{d}z}, \quad C_j(z_i) = \begin{cases} 1, & i=j \\ 0, & i \neq j. \end{cases} \tag{23}$$

The associated quadrature weights are

$$w_j = 2\Big/\left(N(N-1)\{P_{N-1}(z_j)\}^2\right) \tag{24}$$

and the Legendre polynomials can be evaluated by the three-term recurrence $P_0 = 1$, $P_1 = z$, $(n+1)P_{n+1}(z) = (2n+1)zP_n - nP_{n-1}$.

The matrix of grid point values of the first derivative of the cardinal functions is given by

$$\frac{\mathrm{d}C_j}{\mathrm{d}z}(z_i) \equiv \delta_{ij}^{(1)} = \begin{cases} -(1/4)N(N-1), & i=j=0, \\ (1/4)N(N-1), & i=j=N-1, \\ 0, & i=j \text{ and } 0<j<N-1, \\ P_{N-1}(z_i)/[P_{N-1}(z_j)(z_i-z_j)], & i\neq j. \end{cases} \tag{25}$$

The matrix of second derivatives, $\vec{\vec{\delta}}^{(2)}$, is just the product of the first derivative matrix with itself.

We then write

$$u(z) = \sum_{j=0}^{N-1} u_j C_j(z), \tag{26}$$

where the coefficients of the nodal series, u_j, are also the values of the approximation at the collocation points. It then follows that $u_z(z_i) = \sum_{j=0}^{N-1} \delta_{ij}^{(1)} u_j$ and similarly for higher derivatives. The convolution operator is discretized by Gaussian quadrature as

$$\mathscr{D}u(z_i) = \frac{1}{2}\sum_{j=0}^{N-1}\left\{K\left(\frac{\pi}{2}(z_i - z_j)\right) + K\left(\frac{\pi}{2}(z_i + z_j - 2)\right)\right\} w_j u_j = \sum_{j=0}^{N-1} K_{ij} u_j. \tag{27}$$

The algebraic system of $(N+1)$ equations includes $(N-2)$ conditions that are the vanishing of the residual of the integro-differential equation at the $(N-2)$ interior points

$$R_j \equiv (u_j - c)u_{j,XX} + (u_{j,X})^2 - \frac{\pi^2}{4}\sum_{j=0}^{N-1} K_{ij} u_j = 0. \tag{28}$$

The boundary condition $u(0) = c$ is $u_0 = c$ in discrete form. The boundary condition $u_X(X = \pi) = 0$ is $u_X(z_{N-1}) = \sum_{j=0}^{N-1} \delta_{N-1,j}^{(1)} u_j = 0$. The third boundary condition is $\sum_{j=0}^{N-1} K_{0,j} u_j = [u_X(z_0)]^2$. The set of $N+1$ algebraic equations – $(N-2)$ conditions from the differential equation plus the three boundary conditions – is then solved for the $N+1$ unknowns comprising the N grid point values u_j plus the phase speed c.

The standard algorithm for sets of nonlinear algebraic equations is the Newton–Raphson iteration described in all elementary numerical analysis texts such as Appendix C of [7]. Like all iterations, Newton's algorithm requires an initialization or "first guess", and the iteration may fail if the first guess is too far from the desired root. A popular strategy for generating first guesses for a certain dispersion operator $\mathscr{D}$ is to inflate the problem by introducing an artificial parameter λ, and replacing the dispersion operator by

$$\mathscr{D} \to \lambda\mathscr{D} + (1-\lambda)\mathscr{D}^0, \quad \lambda \in [0,1], \tag{29}$$

where $\mathscr{D}^0$ is one of the dispersion operators for which an explicit solution is known. For $\lambda = 0$, the explicit solution for the "base" operator $\mathscr{D}^0$ is the exact solution. The solution for $\lambda = 0$ furnishes the first guess for $\lambda = \mu$ for some $\mu \ll 1$. The solution for $\lambda = \mu$ is a good first guess for $\lambda = 2\mu$, and so on. By preceding in sufficiently small steps in λ, one can bootstrap from $\lambda = 0$ to $\lambda = 1$, each root furnishing a good initialization for the next value of λ, until the target dispersion operator is applied when $\lambda = 1$. A table of such solutions has been given above; the simplest choice is Ostrovsky–Hunter equation, which is $\mathscr{D}^0 \equiv u$.

Numerical solutions need not have zero mean, i.e., $\int_0^\pi u(X)\,\mathrm{d}X = 0$, because of the freedom, when the mean of the kernel of the convolution is zero, to add a constant μ to both c and u simultaneously. We can enforce the zero mean condition by the optional post-processing step

$$\tilde{u}_{\text{mean}} = \sum_{j=0}^{N-1} u_j w_j, \quad u = \tilde{u} - \tilde{u}_{\text{mean}}, \quad c = \tilde{c} - \tilde{u}_{\text{mean}}, \tag{30}$$

where $\tilde{u}$ and $\tilde{c}$ are the grid point values and phase speed of the solution before the post-processing step, and u and c are the corresponding values for the zero mean solution, and the w_j are the Legendre–Lobatto quadrature weights.

Our numerical treatment of the convolution integral implicitly assumes that the kernel $K(X)$ is an analytic function. However, the wave equation is still well-posed even if K has delta-function singularities or discontinuous slopes. We are unapologetic about presenting an algorithm only for smooth K. The kernel must contain delta-functions for the Korteweg-deVries and Fifth-Order Korteweg-deVries equations, but these do not have corner waves, and their travelling wave solutions can be found by attacking the differential equation, rather than the integro-differential form. Whitham [30,31] has discussed kernels which have slope discontinuities at $X = 0$. We explain in a later section that these usually arise when the kernel is a Green's function, and Whitham devised a simple procedure for converting such integro-differential equations back into ordinary differential equations. Thus, there has been little interest in solving Whitham's Equation Family in integro-differential equation form for non-smooth kernels.

If $K(X-y)$ had a slope discontinuity at $X = y$, one could recover spectral accuracy by writing the integral as the sum of two separate integrals, one on $y \in [0, X]$ and the other on $y \in [X, 2\pi]$, with smooth kernels on new integration intervals. Implementing Gaussian quadrature is straightforward but messy, so the details are not given here.

4. Numerical example

Fig. 3 shows how the Gabov–Shefter–Rosales problem, which is the dispersion operator with the kernel $K(x) = \pi\cos(x)$, can be solved by continuation from the known solution for Ostrovsky–Hunter equation.

Fig. 4 shows that the numerical difficulties created by the slope discontinuity at $X = 0$ have been completely eliminated: the error falls roughly linearly on a log-linear plot of error versus the number of grid points N. Because the exact solution, restricted to the interval $X \in [0, \pi]$, is a constant plus a term proportional to $\sin(X/2) = \sin((\pi/4)(z+1))$, a so-called entire function with no singularities except at infinity in the complex z-plane, the convergence is extremely fast. The maximum pointwise error in $u(X)$ is almost identical with the error in the phase speed; with just 15 grid points, both errors are only about 4.3×10^{-12}!

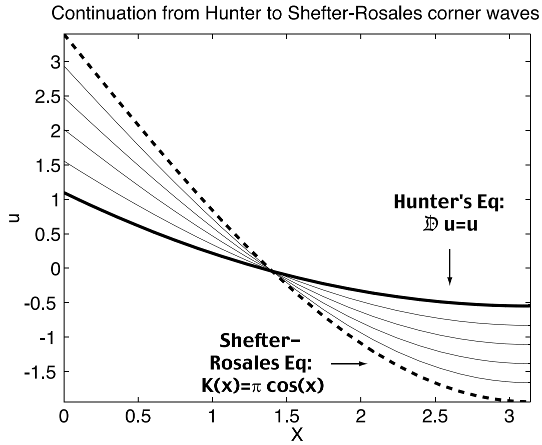

Fig. 3. Continuation from the corner wave of Ostrovsky–Hunter equation (thick solid curve) to the corner wave of the Gabov–Shefter–Rosales equation (thick dashed curve); several intermediates are also shown.

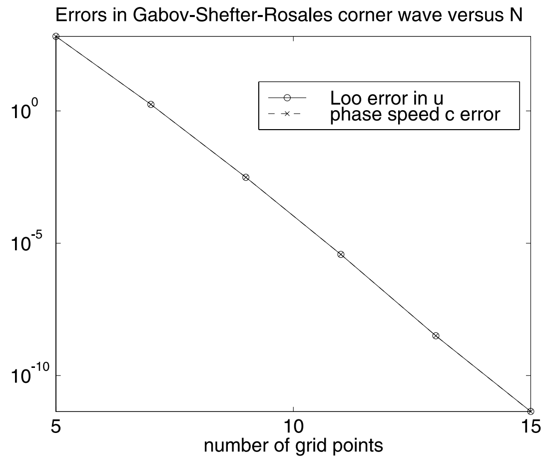

Fig. 4. Maximum pointwise error (L_∞ error) is shown as the circles and the error in c is shown as the $\times$s, both plotted versus N, which is simultaneously the number of collocation points and also the number of Legendre polynomials in the truncated spectral basis. The maximum error in u is graphically indistinguishable from the corresponding error in c, although there are slight numerical differences between them, so the circles and $\times$s are are almost superimposed.

5. Known explicit solutions

Explicit solutions are valuable as starting points for the continuation method described in the next section and as test problems. Since these explicit solutions have been published only in scattered places, it is useful to collect known explicit solutions as Table 1.

6. Greens function kernels

Whitham [30,31] and Naumkin and Shishmarev [20] in their monograph chose to write the wave equation for travelling waves as, with $v(X) \equiv u(X) - c$ as before,

Table 1
Explicit corner waves for special cases of Whitham's equation family

Equation name	Dispersion operator $\mathscr{D}u$	Corner wave u, c	Sources
Ostrovsky–Hunter	u	$\frac{\pi^2}{9} - \frac{\pi}{3}\lvert x\rvert + \frac{1}{6}x^2, \quad c = \pi^2/9$	[16,17]
Gabov/Shefter–Rosales	$\int_0^{2\pi} \cos(x-y)u(y)\,\mathrm{d}y$	$\frac{32}{3\pi} - \frac{16}{3}\sin(\lvert X\rvert/2), \quad c = 32/(3\pi)$	[15,26]
Whitham	$pb^2\left\{u - \int_0^{2\pi} \frac{b\cosh(b\{\lvert X-y\rvert - \pi\})}{2\sinh(\pi b)}u(y)\,\mathrm{d}y\right\}$	$-\frac{4}{3}p\left\{1 - \frac{\cosh([b/2](X-\pi))}{\cosh([b/2]\pi)}\right\} + c,$ $c = \frac{4}{3}p\left\{1 - \frac{2}{\pi b}\tanh\left(\frac{\pi b}{2}\right)\right\}$	[25,30–32]

$$vv_X = -p\int_0^{2\pi} G(X-y)v_y(y)\,\mathrm{d}y, \tag{31}$$

where p is a constant. By an integration-by-parts, one can show that Whitham's form is equivalent to the convolution integral form used here (5) if the kernel $K(X) = -\pi p G_{XX}$.

Whitham observed that if G is the Green's function for a linear, constant coefficient differential operator $\mathscr{L}$, i.e.,

$$\mathscr{L}G = \delta(x), \tag{32}$$

where δ is the usual Dirac delta-function, then the *integro-differential* equation can be converted into a more easily soluble *differential* equation. If the differential operator associated with G is applied to both sides of the integro-differential equation, the wave equation becomes

$$\mathscr{L}\{vv_X\} = -p\int_0^{2\pi} \mathscr{L}\{G(X-y)\}v_y(y)\,\mathrm{d}y = -pv_X \tag{33}$$

because the integration of the product of a function with the delta-function is just the value of the factor of δ, evaluated at $X = y$. The Legendre-pseudospectral algorithm can be applied to the differential equation just as to the integro-differential equation.

Since the corner waves for Whitham's own particular choice of a Green's function are explicitly known and also because there are no particular numerical novelties in solving a problem that is purely a differential equation, we have not worked out a numerical example.

7. Summary and open questions

As noted earlier, a motive in deriving an algorithm to compute corner waves is that these arise in a wide variety of wave systems, not merely Whitham's family of equations. Equatorially trapped Kelvin waves were described in Section 1. The Camassa–Holm equation also has solitary waves with slope discontinuities ("peakons") and spatially periodic waves known as "coshoidal waves" [5,10,11]. Similar structures known as "compactons" are described in [14,24]. Since compactons and peakons are also known in explicit form, there is no need for numerical methods to compute them. However, when these wave equations are generalized, explicit solutions are no longer possible, and the Legendre-pseudospectral method is useful.

On the theoretical side, Naumkin and Shishmarev have made considerable progress in proving existence theorems and breaking-sufficiency theorems in a series of papers summarized in their book [20]. However, a sample of theorems-desired-but-not-yet-proved includes

1. a proof of the existence of corner waves for a broad class of dispersion operator;
2. a proof that corner waves are symmetric;
3. a proof that corner waves are attractors for all time-dependent solutions of sufficiently large initial energy.

In his 1967 paper, Whitham also discussed a dispersion operator in the form (for the *infinite* interval)

$$\mathscr{D} = -\int_{-\infty}^{\infty} K_g(X-y)u_y \, \mathrm{d}y, \tag{34}$$

where the kernel is the Fourier transform of the dispersion relation for linear water waves:

$$K_g = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(\mathrm{i}kX)\sqrt{\frac{g}{k}\tanh(kH)}\, \mathrm{d}k. \tag{35}$$

Although the transform cannot be done analytically so as to give K_g in explicit form, it is known that the kernel is *singular* at $X = 0$ as $1/\sqrt{X}$. The algorithm described here, which implicitly assumes that $u(X)$ and $K(X)$ are well-behaved functions, will not be exponentially convergent for such a kernel. Furthermore, the solution itself becomes "cusped with a vertical tangent" [30, p. 23]. There are various techniques for coping with such singularities in pseudospectral algorithms [7], but we have not attempted such an extension here: a wave of infinite slope at the crest is not of great physical or engineering interest.

For non-singular dispersion operators and corner waves of finite slope, the principle numerical challenge is to extend the computational algorithm to more than one space dimension and to more complicated differential equations, as exemplified by the equatorially trapped Kelvin waves described in the introduction. This in turn is tied to knotty theoretical questions: Does the limiting Kelvin wave truly have a slope discontinuity, or is this an illusion created by the difficulty of accurate numerical solutions when the curvature is very high at the crest? If the slope discontinuity is real, does it extend to all latitudes, or does the Kelvin wave become smooth at sufficiently high latitudes where the wave has decayed to small amplitude?

Although the Legendre-pseudospectral algorithm converges exponentially fast for Whitham's equation family with smooth dispersion operators and is easy to program, we have taken only a few toddler steps towards the goal of understanding corner waves and wave equations with finite amplitude breaking.

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